QIS/QCS Seminar

Meeting 1

Alex Heilman

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Overview

- Qubits and in general, qudits

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Qubits and Qudits

Only dealing with a finite dimensional, discrete system (in contrast to the continuous states of position and momentum).

Qubit state: $$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \qquad \alpha\alpha^* + \beta\beta^* = 1$$

Qudit state: $$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \vdots \end{bmatrix} \qquad \alpha\alpha^* + \beta\beta^* + \gamma\gamma^* + ... = 1$$

Computational Basis

We now define some specific orthonormal basis, which we will term the *computational basis*, taking the form:

$$|i\rangle = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ d \end{matrix}$$

which clearly satisfies

$$\langle i|j\rangle = \delta_{ij}$$

and where we have used the common notation $\langle\psi| = [\psi_1^* \ \psi_2^* \ \ldots]$

Aside: Bloch Sphere

Elements of the qubit state space may be parametrized by three angles, as below:

$$|\psi\rangle = e^{i\gamma}(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle)$$

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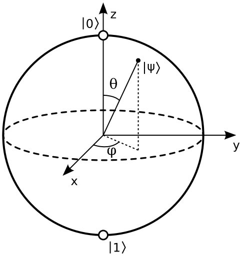

Interlude: Proto-typical Examples

Qubits:

- Electron Spin (Like Stern-Gerlach)
- Photon Polarization
- Symmetric/Antisymmetric Electron Pair (Singlet $\leftrightarrow$ Triplet)

Qudits:

- Energy Levels

Postulates of QM (Vectors)

1. The state space of isolated physical systems is a Hilbert space. The state of the system is completely determined by a vector in this space.

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4. The initial state of a composite system consisting of several initial substates is the tensor product of all those initial substates.

Inner Product Space I

An inner product space is a vector space V equipped with a binary product $(\cdot, \cdot)$ between elements of the vector space $|\psi\rangle \in V$ that satisfies the following requirements:

- Linear in the one of the arguments

$$\left(|u\rangle, \sum_i \lambda_i |v_i\rangle\right) = \sum_i \lambda_i (|u\rangle, |v_i\rangle)$$

- Conjugate symmetric under exchange

$$(|u\rangle, |v\rangle) = (|v\rangle, |u\rangle)^*$$

- Positive-definite for non-zero vectors

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In finite dimensions, a complex inner product space is equivalent to a Hilbert space.

Inner Product Space II

In our case where $|u\rangle, |v\rangle \in \mathbb{C}^n$, we may define the inner product:

$$(|u\rangle, |v\rangle) = \sum_i u_i^* v_i = \begin{bmatrix} u_1^* & u_2^*, \dots u_n^* \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

which we will denote in the bra-ket notation to be:

$$\langle u|v\rangle$$

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The definition of an inner product, as above, allows us to define a norm on the vector space:

$$\| \; |u\rangle \; \| = \sqrt{\langle u|u\rangle}$$

Inner Product Space III

The Cauchy-Schwarz Inequality: The inner product in an inner product space is guaranteed to satisfy the following relation:

$$|\langle u|v\rangle|^2 \leq \langle u|u\rangle\langle v|v\rangle$$

this is often useful, and more general than the triangle inequality.

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The inner product and structure of bra-ket notation allows us to express the action of linear operators on the space in a certain basis $\{|i\rangle\}$ in a useful way, by utilizing an outer product representation.

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In the case that this basis coincides with the eigenbasis of the operator, we essentially have the spectral decomposition/diagonal form of U.

Inner Product Space IV

Example: The outer product of two two-by-one vectors gives back a matrix. As an explicit example, consider the two vectors below, and their outer product.

$$|a\rangle = \begin{bmatrix} 3 \\ 13 \end{bmatrix}, \ |b\rangle \begin{bmatrix} 7i \\ 17 \end{bmatrix} \qquad |a\rangle\langle b| = \begin{bmatrix} 3 \\ 13 \end{bmatrix} \begin{bmatrix} -7i & 17 \end{bmatrix} = \begin{bmatrix} -21i & 51 \\ -91i & 241 \end{bmatrix}$$

Note that in the special case of computational basis elements, each outer product corresponds to one parameter in the matrix; with the further special case of the outer product of one basis element with itself being an element on the diagonal.

$$a_{10}|1\rangle\langle 0| = \begin{bmatrix} 0 & a_{10} \\ 0 & 0 \end{bmatrix} \qquad a_{00}|0\rangle\langle 0| + a_{11}|1\rangle\langle 1| = \begin{bmatrix} a_{00} & 0 \\ 0 & a_{11} \end{bmatrix}$$

Inner Product Space V

Spectral Decomposition: A very important fact about normal and or real symmetric matrices (satisfying $[M, M^*] = 0$) follows from the more general spectral theorem. All Hermitians and unitaries are representable in some basis (formed by their eigenvectors) in which they are diagonal. Explicitly, we have for such matrices:

$$M = \sum_{i=1}^{D} \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

where M is a normal $D \times D$ matrix and $|\lambda_i\rangle$ refers to M's i-th normalized eigenvector with eigenvalue λ_i. Note that Hermitians are guaranteed to have all real eigenvalues λ_i.

This is equivalent to saying these matrices' eigenvectors form an orthonormal basis.

Why a Hilbert Space?

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This, of course can't be entirely answered. However, at a most basic level, physical theories describe how we may describe our expectations of results of measurements of certain systems in time (see QBism). In essence, we may describe these prospective measurements and their outcomes as an algebra of observables on the state space. And, due to the Gelfand-Naimark theorem, this algebra of observables may be represented/realized as a set of operators on some Hilbert space

Pure States and Mixed States/Density Matrices

Pure states are normalized rays in Hilbert space, representable simply as vectors or kets of the form $|\psi\rangle$, or equivalently as positive operators $|\psi\rangle\langle\psi|$.

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Operators satisfying the above constitute a representation of states known as density matrices or density operators

Pure States and Ensembles/Density Matrices

Fidelity: While the 'closeness' of two pure states is readily describable via their inner product, we need to define some measure for the 'closeness' of density matrices. One common measure for 'closeness' is fidelity, defined below

$$F(\rho, \sigma) = \mathrm{Tr}(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})$$

which is symmetric, and bounded within the range $0 \leq F \leq 1$.

Purification: Purification allows us to describe any mixed state as a pure state in a larger space. Explicitly, we may always find a state $|AR\rangle$ such that $\rho_A = \mathrm{Tr}_R(|AR\rangle)$ for arbitrary ρ_A

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• Ensembles of states, like regular probability distribution of states
Simply sum over set of states in distribution, scaled by respective probability.

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• Substate description, simply take partial trace over unwanted subsystem

$$\rho_A = \mathrm{Tr}_B(\rho_{AB})$$

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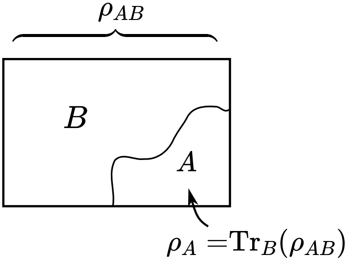

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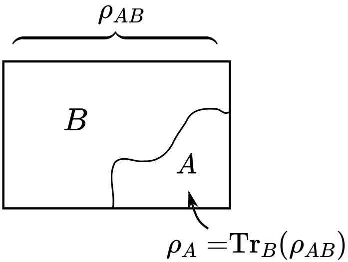

The resulting state is termed the *reduced density matrix* and describes our state of knowledge of the subsystem. This reduced state is also interpretable as that left over after averaging over all measurements on the forgotten subspace.

Partial Trace II

For separable states, the partial trace takes the simple form:

$$\rho_A = \mathrm{Tr}_B(\rho_A \otimes \rho_B) = \rho_A \otimes \mathrm{Tr}(\rho_B)$$

Example As a simple example, consider the two qubit density matrix:

$$\rho = \frac{1}{2}\left(|00\rangle\langle 00| + |10\rangle\langle 10|\right)$$

Now, for reasons that will be conducive to the example, we may rewrite the above state as the tensor product state:

$$\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)_1 \otimes (|0\rangle\langle 0|)_2$$

with the tensor product structure explicitly labeled as subscripts. Now, we take the partial trace over the first qubit's state space, as follows:

$$\mathrm{Tr}_1(\rho) = \langle 0|_1 \rho |0\rangle_1 + \langle 1|_1 \rho |1\rangle_1 = |0\rangle\langle 0|$$

where we may now omit the subscript in the output since the result is a one-qubit state.

Note that the state need not be separable to perform the partial trace.

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Essentially, the partial trace is the unique operation that preserves the relevant, expected measurement statistics of subsystems.

Time Evolution

Basic quantum physics tells us that the time derivative of an isolated state (in the Schrodinger picture) is of the following form:

$$\frac{d|\psi\rangle}{dt} = \frac{1}{i\hbar} H|\psi\rangle$$

where H is the Hamiltonian of the system, and which is Hermitian. We often simplify the above by choosing $\hbar = 1$.

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This has a solution of the form below,

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$$

Unitary Transformations

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Thus, as in Postulate 2: The evolution of any state over a finite period of time is describable by a unitary transformation U:

$$|\psi(t_2)\rangle = U(t_1, t_2)|\psi(t_1)\rangle$$

where $U(t_1, t_2) = e^{-iH(t_2 - t_1)/\hbar}$. (Like the time evolution operator, which may be solved for perturbatively using Dyson series/Feynman diagrams)

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Similarly, an arbitrary density operator evolves as:

$$|\psi(t_2)\rangle\langle\psi(t_2)| = U(t_1, t_2)|\psi(t_1)\rangle\langle\psi(t_1)|U^\dagger(t_1, t_2)$$
$$\Rightarrow \rho \rightarrow U\rho U^\dagger$$

Measurement in Detail

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POVM are equivalent to projective measurements on a larger space along with unitary transformations (see Naimark's Dilation Theorem). They're like the 'mixed states of measurements'.

A measurement of a quantum system is most generally described as a set of measurement operators $\{M_m\}$ (with m denoting a possible measurement outcome m) that satisfy the completion relation:

$$\sum_m M_m^\dagger M_m = \mathbb{I}$$

with measurement outcome m occuring after measuring state $|\psi\rangle$ and yielding state $|\psi_m\rangle$ with probability $p(m)$.

$$|\psi_m\rangle = \frac{1}{\sqrt{p(m)}} M_m |\psi\rangle \qquad w/ \qquad p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$$

Note that the completeness relation enforces that all our possible outcomes' probabilities sum to one.

Projective measurements are described by some Hermitian operator, decomposable (via the Spectral decomposition theorem) as:

$$M = \sum_{m=1}^{d} \lambda_m P_m$$

where P_m is a projection operator onto the eigenspace of M corresponding to eigenvalue λ_m.

Projection Operators: Projection operators are Hermitian operators that project states onto their subspaces. Explicity, for some n-dimensional subspace k spanned by the (orthonormal) basis $\{|i\rangle\}$, projection operators have the form:

$$P_k = \sum_{i=1}^{n} |i\rangle\langle i|$$

Application of the measurement operator results in output state corresponding to measurement outcome m with probability p_m, as below:

$$|\psi\rangle \to |\psi_m\rangle = \frac{P_m|\psi\rangle}{\sqrt{p_m}} \qquad w/ \;\; p_m = \langle\psi|P_m|\psi\rangle$$

For density matrices, we then have the post-measurement state:

$$\frac{P_m \rho P_m^\dagger}{\mathrm{Tr}\left(P_m \rho P_m^\dagger\right)} \qquad w/ \;\; p_m = \mathrm{Tr}\left(P_m \rho P_m^\dagger\right)$$

POVMs describe a more general picture of measurements (i.e. lack of repeatability, inconclusive results, etc.)

Next time!

Interacting/Simultaneous Systems I

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As in Postulate 4: States of composite systems, where the i-th subsystem is know to be in state $|\psi_i\rangle$ are described jointly by the tensor product of all the substates

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes .. \otimes |\psi_n\rangle$$

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In the language of density operators: for a composite system where each substate is known to be in state ρ_i, the total state is describable as the tensor product of them

$$\rho_1 \otimes \rho_2 \otimes ... \otimes \rho_n$$

Interacting/Simultaneous Systems II

While not all states can be written as a tensor product of states in the respective subsystem, the total state space is the tensor product space of the two Hilbert spaces.

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

States that are representable as two substates tensored together are termed separable. This doesn't have a clear/easy criteria by which to determine whether a given state is separable (except in two-qubit spaces by the Peres–Horodecki criterion).

Non-separable states are considered entangled, but a true 'measure' of entanglement in general is still nebulous.

Interacting/Simultaneous Systems III

Example As a simple example of the tensor product, consider the tensor product of the two 2×2 matrices below:

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \otimes \begin{bmatrix} 7 & 11 \\ 13 & 17 \end{bmatrix} = \begin{bmatrix} 1 \cdot \begin{bmatrix} 7 & 11 \\ 13 & 17 \end{bmatrix} & 2 \cdot \begin{bmatrix} 7 & 11 \\ 13 & 17 \end{bmatrix} \\ 3 \cdot \begin{bmatrix} 7 & 11 \\ 13 & 17 \end{bmatrix} & 5 \cdot \begin{bmatrix} 7 & 11 \\ 13 & 17 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 7 & 11 & 14 & 22 \\ 13 & 17 & 26 & 34 \\ 21 & 33 & 35 & 55 \\ 39 & 51 & 65 & 85 \end{bmatrix}$$

We also may take the tensor product of vectors, as in the case of pure states. As an example, consider the case below:

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \cdot \begin{bmatrix} 5 \\ 7 \end{bmatrix} \\ 3 \cdot \begin{bmatrix} 5 \\ 7 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 15 \\ 21 \end{bmatrix}$$

Note that not every four-by-one vector is representable as the tensor product of two two-by-one vectors. As an example, consider the bell state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$:

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \neq |\psi_1\rangle \otimes |\psi_2\rangle \quad \forall\ |\psi_1\rangle, |\psi_2\rangle$$

Note that the matrix form of the tensor product given in the example is also referred to as a Kronecker product or matrix direct product.

Why the Tensor Product?

The tensor product is the natural choice for the packaging of related states since we may act on subsystems individually by taking the tensor product of our subsytem's operator and identity in the rest.

$$|\phi\rangle \otimes |\psi\rangle \quad \longrightarrow \quad (\mathbb{I} \otimes U)(|\phi\rangle \otimes |\psi\rangle)$$

$$\downarrow \ \mathrm{Tr}_\phi$$

$$|\psi\rangle \quad \longrightarrow \quad U|\psi\rangle$$

General Evolution I

We may now construct a more general picture of arbitrary quantum time evolution that takes density matrices to density matrices.

General Evolution II

Imagine we have some small system we're interested in, but it inevitably interacts with some larger system we'll term the environment.

$$\rho \to \rho \otimes \rho_{Env.}$$

It then evolves in concert with this larger system according to a unitary transformation:

$$\rho \otimes \rho_{Env.} \to U(\rho \otimes \rho_{Env.})U^\dagger$$

However, we still only care about and measure the subsystem, hence we end up with a reduced density matrix:

$$U(\rho \otimes \rho_{Env.})U^\dagger \to \text{Tr}_{Env.}\left[U(\rho \otimes \rho_{Env.})U^\dagger\right]$$

General Evolution III

So, in total, a more general map of evolution follows a form similar to that below:

$$\rho \to \rho \otimes \rho_{Env.} \to U(\rho \otimes \rho_{Env.})U^\dagger \to \mathrm{Tr}_{Env.}\left[U(\rho \otimes \rho_{Env.})U^\dagger\right] = \rho'$$

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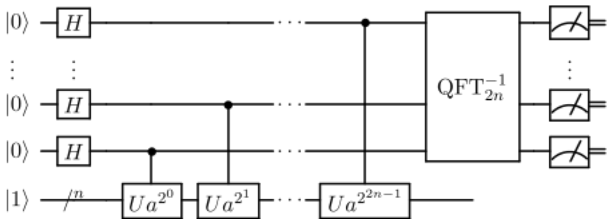
The most general maps that returns arbitrary quantum states (and act on density operators) are those of completely-positive linear maps.

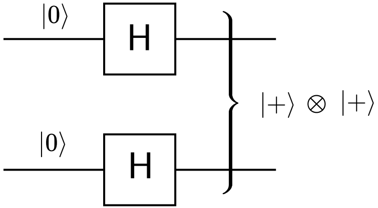
Quantum Circuit Model

Inspired by electronic circuits, we may define many quantum algorithms in terms of neat diagrams describing a system's state evolution and measurements.



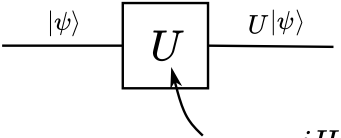
Wires/Initialization

Each wire in a quantum circuit represents a qubit/qudit. They're often taken to be initialized in some pure state $|0\rangle$ and tensored together.



Gates from Hamiltonians

Quantum computation often takes advantage of a set of *gates* to construct algorithms. Schrodinger's equation tells us the form of the state's time derivative (Schrodinger picture).

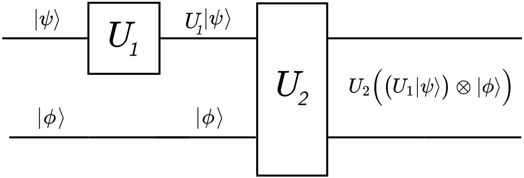


$$= e^{-iH\Delta t}$$

So, for each gate, we need a corresponding Hamiltonian that will generate it, and a period of time over which the Hamiltonian's action will coincide with the desired unitary action.

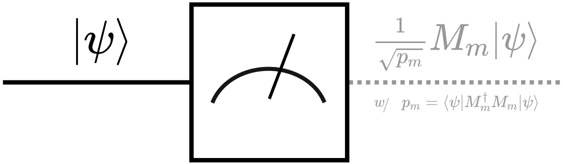
Localized Gates

Gates only act on the state space of the qubits contained within their diagramatic box, acting as identity on all the other qubits.



In the example above, U_1 acts only on the first qubit, so it acts as $U_1 \otimes \mathbb{I}$ on the total statespace; whereas U_2 acts on both qubits and hence may not even be decomposable as a tensor product of two local gates. U_1 then would be said to be a 'local' gate, acting locally only on the first qubit.

Measurement



Measurement may then be performed on each qubit, as desired, with the convention being that measurements are performed in the computational basis

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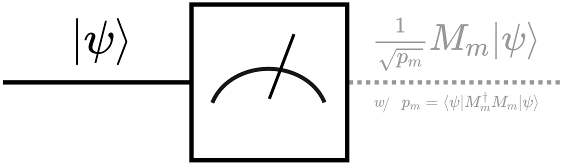

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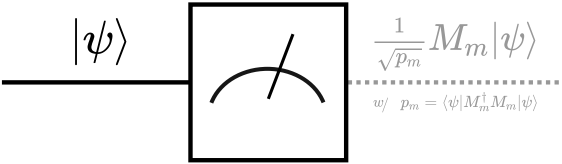

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More often than not in the circuit model, the output state is discarded and only the measurement outcome is treated.

Recap

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- Consider more general set of quantum operations and measurements, describes quantum channels and gives potential model for noise
- Package the more basic elements into neat 'circuits'

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- Qubits and qudits, really just finite-dimensional QM
- Pure states and ensembles, how to transfer between them and advantages of density operator formalism
- Consider more general set of quantum operations and measurements, describes quantum channels and gives potential model for noise
- Package the more basic elements into neat 'circuits', allows for succinct description of quantum algorithms

Next time

- What to look for in prospective qubit realizations
- Basic physics of some real-world examples