An Example of an Entangled Mixed State with Classical Correlations

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Revised: 1/18/2022

Abstract

Entanglement seems to be an operational feature unique to quantum systems. In this paper, a specific example of a definitively entangled state will be shown to exhibit purely classical correlations. The state is defined as a mixed state parametrized on a path through two qubit space connecting the unpolarized state with the EPR state. For a range of values of this parametrization, it will be shown, by the Peres-Horodecki condition, to be entangled. An analytic expression for the joint probability distribtution of events corresponding to measurement with Hermitian operators will be constructed. This will then be shown to be classical for a range of the parameterization that overlaps with the entangled range.

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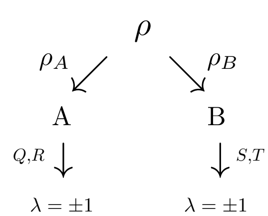
1 Introduction

In 1989 Reinhard F. Werner published a paper[1] with a concrete example of a quantum state that is entangled yet acts classically. We use the specific parametrized state, ρ_w, to show that states can be entangled and yet produce classical correlations regardless of the local projective measurements chosen.

$$\rho_w = p\,|s\rangle\,\langle s| + (1-p)\frac{I}{4} = \begin{bmatrix} \frac{1-p}{4} & 0 & 0 & 0 \\ 0 & \frac{1+p}{4} & \frac{-p}{2} & 0 \\ 0 & \frac{-p}{2} & \frac{1+p}{4} & 0 \\ 0 & 0 & 0 & \frac{1-p}{4} \end{bmatrix}_{Std.Basis}$$

We consider this state in the framework of a two party model, where two separated persons make a local measurement on the system simultaneously. Each person is allowed choice between two prospective measurements, which all have two outcomes.



This small model size is defend-able as it is the basic setting for Bell inequalities. Thus, it is a large enough framework to observe extra-classical correlations, yet small enough to be manageable. This is an interesting investigation because entanglement is one of the fundamental properties of quantum states responsible for non-classical behaviors, and such an investigation shows entangled states are not necessarily non-classical.

2 Preliminaries

Projective Measurements

Projective measurements on quantum states are described by Hermitian operators. For two dimensional quantum states, called qubits, the Paulis $\sigma_{0,1,2,3}$ (a well known class of two by two matrices) form a basis.

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus, all local projective measurements, U, on a qubit system can be written in terms of the Paulis as such:

$$U = \vec{u} \cdot \vec{\sigma} = u_1\sigma_1 + u_2\sigma_2 + u_3\sigma_3 = \begin{bmatrix} u_3 & u_1 - iu_2 \\ u_1 + iu_2 & -u_3 \end{bmatrix}$$

Where $\vec{u}$ is a unit vector. These local measurements are then simply tensored together to simultaneously act on the entire system from potentially distant locations.

$$U \otimes V \longrightarrow \textit{simultaneously measure } U \textit{ and } V$$

Projectors are matrices that project the vector space of the state onto the space of a certain outcome of the measurement operator. For the Pauli matrices, they are simply constructed as:

$$\mathbb{P}_+ = \frac{1 + \vec{u} \cdot \sigma}{2} \qquad \mathbb{P}_- = \frac{1 - \vec{u} \cdot \sigma}{2}$$

Here $\mathbb{P}_+$ is the projector onto the $\lambda = +1$ eigenspace of U and $\mathbb{P}_-$ is the projector onto the $\lambda = -1$ eigenspace of U.

Probability of Outcome

The probability of a certain measurement or set of measurements, M, on a density matrix, ρ, yielding a certain outcome, O, is given by the expression:

$$P_{M \to O} = tr(\mathbb{M}_O \cdot \rho) \tag{1}$$

Here, $\mathbb{M}_O$ is the projector of measurement M associated with outcome O. For local projective measurements written in terms of the Paulis, the outcomes will be labeled $+1$ and -1, the eigenvalues of the measurement operator (as the projector will project onto the eigenspaces associated with these λ's). The probability of n simultaneous local projective measurements is calculated by substituting $\mathbb{P}_\pm^{\otimes n}$ for $\mathbb{M}_\mathbb{O}$.

Local Hidden Variables

A global section, as defined by Abramsky[2], is analogous to a joint probability distribution, in that it describes the 'quasi-probability' of every prospective measurement giving an associated set of outcomes, even if they cannot be performed simultaneously. For our model, it would describe the probability of Q,S,R, and T each going to some corresponding outcome ($P_{QRST\to\pm1\pm1\pm1\pm1}$), an example of which will be given for the Werner state in Section 3.3. The requirements of a global section are that all the values sum to one and they marginalize correctly to the observed probabilities. All quantum models have a global section over the reals, meaning these quasi-probabilities exist, but they may require the use of negative entries. Global sections that do not require negative values may be described with a local hidden variable, and correspond to a scenario that is classically correlated.

Separability

Separable states are density matrices of the form:

$$\rho = \sum_i p_i \rho_{Ai} \otimes \rho_{Bi} \tag{2}$$

Thus, they are made by taking the kronecker product of density matrices of smaller systems, and then summing over a probability distribution on them. So,

$$\textstyle\sum_i p_i = 1$$

While verifying if a density matrix is separable or entangled is generally a very hard problem, there thankfully exists an if and only if condition for small systems. This is known as the Peres-Horodecki condition, or Positive Partial Transpose (PPT), which requires that the partial transpose of the density matrix be positive. Partial transpose of a 4×4 matrix follows:

$$\left(\begin{array}{cc} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \begin{bmatrix} c_{00,00} & \swarrow c_{00,01} & c_{00,10} & \swarrow c_{00,11} \\ c_{01,00}\nearrow & c_{01,01} & c_{01,10}\nearrow & c_{01,11} \\ c_{10,00} & \swarrow c_{10,01} & c_{10,10} & \swarrow c_{10,11} \\ c_{11,00}\nearrow & c_{11,01} & c_{11,10}\nearrow & c_{11,11} \end{bmatrix} \end{array} \right) \xrightarrow{I\otimes T} \left(\begin{bmatrix} c_{00,00} & c_{01,00} & c_{00,10} & c_{01,10} \\ c_{00,01} & c_{01,01} & c_{00,11} & c_{01,11} \\ c_{10,00} & c_{11,00} & c_{10,10} & c_{11,10} \\ c_{10,01} & c_{11,01} & c_{10,11} & c_{11,11} \end{bmatrix} \right)$$

Positivity of the partial transpose can be determined by solving for eigenvalues of ρ^{pT} and checking if they are non-negative. If the eigen-values are non-negative the state is deemed separable and, therefore, un-entangled (only for small systems, such as ours). If one or more of the partial transposes eigenvalues are negative, it is deemed entangled.

3 State Properties

3.1 Entanglement and Separability

We test the Werner state for separability via the PPT condition outlined in Section (REMOVE *'s?). Taking the partial transpose of ρ_W:

$$\left(\begin{bmatrix} \frac{1-p}{4} & 0 & 0 & 0 \\ 0 & \frac{1+p}{4} & \frac{-p}{2} & 0 \\ 0 & \frac{-p}{2} & \frac{1+p}{4} & 0 \\ 0 & 0 & 0 & \frac{1-p}{4} \end{bmatrix}\right) \xrightarrow{I\otimes T} \begin{bmatrix} \frac{1-p}{4} & 0 & 0 & \frac{-p}{2} \\ 0 & \frac{1+p}{4} & 0 & 0 \\ 0 & 0 & \frac{1+p}{4} & 0 \\ \frac{-p}{2} & 0 & 0 & \frac{1-p}{4} \end{bmatrix}$$

And solving for the eigenvalues of ρ_W^{pT},

$$det\left(\begin{bmatrix} \frac{1-p}{4}-\lambda & 0 & 0 & \frac{-p}{2} \\ 0 & \frac{1+p}{4}-\lambda & 0 & 0 \\ 0 & 0 & \frac{1+p}{4}-\lambda & 0 \\ \frac{-p}{2} & 0 & 0 & \frac{1-p}{4}-\lambda \end{bmatrix}\right) = (\frac{1+p}{4}-\lambda)\begin{vmatrix} \frac{1-p}{4}-\lambda & 0 & \frac{-p}{2} \\ 0 & \frac{1+p}{4}-\lambda & 0 \\ \frac{-p}{2} & 0 & \frac{1-p}{4}-\lambda \end{vmatrix}$$

$$= (\frac{1+p}{4}-\lambda)^2\begin{vmatrix} \frac{1-p}{4}-\lambda & \frac{-p}{2} \\ \frac{-p}{2} & \frac{1-p}{4}-\lambda \end{vmatrix} = (\frac{1+p}{4}-\lambda)^2((\frac{1-p}{4}-\lambda)^2-\frac{p^2}{4}) = 0$$

$$\lambda = \frac{1+p}{4}, \frac{1+p}{4}, \frac{8(1-p)\pm\sqrt{8^2(1-p)^2+4(16)(4p^2-(1-p)^2)}}{32}$$

We see the smallest value of λ is $\frac{8(1-p)-\sqrt{8^2(1-p)^2+4(16)(4p^2-(1-p)^2)}}{32}$, as p can only take the values $[0,1]$. This value of λ is non-negative for $p \leq \frac{1}{3}$. Thus, via the PPT criterion, ρ_W is definitively separable for this range and entangled for $p > \frac{1}{3}$.

3.2 Outcome Probabilities

Using (1) for the Werner State, ρ_w, we get the following formula for probabilities of outcomes with local projective measurements:

$$P_{UV\to wy} = tr(U\otimes V\cdot\rho_w) = \tfrac{1}{4}(1-pwy\chi_{uv}) \tag{3}$$

Here, $\chi_{uv} = \vec{u}\cdot\vec{v} = u_1v_1+u_2v_2+u_3v_3$ and $w,y=\pm1$. This formula is easy to verify:

$$tr\left(\frac{1}{4}\begin{bmatrix} (1+u_3)(1+v_3) & (1+u_3)(v_1-iv_2) & (u_1-iu_2)(1+v_3) & (u_1-iu_2)(v_1-iv_2) \\ (1+u_3)(v_1+iv_2) & (1+u_3)(1-v_3) & (u_1-iu_2)(v_1+iv_2) & (u_1-iu_2)(1-v_3) \\ (u_1+iu_2)(1+v_3) & (u_1+iu_2)(v_1-iv_2) & (1-u_3)(1+v_3) & (1-u_3)(v_1-iv_2) \\ (u_1+iu_2)(v_1+iv_2) & (u_1+iu_2)(1-v_3) & (1-u_3)(v_1+iv_2) & (1-u_3)(1-v_3) \end{bmatrix}\cdot\begin{bmatrix} \frac{1-p}{4} & 0 & 0 & 0 \\ 0 & \frac{1+p}{4} & \frac{-p}{2} & 0 \\ 0 & \frac{-p}{2} & \frac{1+p}{4} & 0 \\ 0 & 0 & 0 & \frac{1-p}{4} \end{bmatrix}\right)$$

$$\begin{aligned} &= \tfrac{1}{16}((1-p)(1+u_3)(1+v_3)+(1+p)(1+u_3)(1-v_3)-2p(u_1-iu_2)(v_1+iv_2) \\ &\quad +(1+p)(1-u_3)(1+v_3)-2p(u_1+iu_2)(v_1-iv_2)+(1-p)(1-u_3)(1-v_3)) \\ &= \tfrac{1}{16}((1-p)(1+u_3+v_3+u_3v_3+1-u_3-v_3+u_3v_3) \\ &\quad +(1+p)(1+u_3-v_3-u_3v_3+1-u_3+v_3-u_3v_3) \\ &\quad -2p(u_1v_1+u_2v_2-iu_1v_2+iu_2v_1+u_1v_1+u_2v_2-iu_2v_1+iu_1v_2)) \\ &= \tfrac{1}{16}(4-4p(u_1v_1+u_2v_2+u_3v_3)) = \tfrac{1}{4}(1-p(u_1v_1+u_2v_2+u_3v_3)) \end{aligned}$$

3.3 Global Section

In a model where each of the two parties is allowed choice between two local measurements (Q or R for party 1, S or T for party 2), the entries of the global section are of the form:

$$P_{QRST\to wxyz} = \frac{1}{16}(1-p(\underbrace{wy\chi_{QS}+wz\chi_{QT}+xy\chi_{RS}+xz\chi_{RT}-wx\chi_{QR}-yz\chi_{ST}}_{(*)}))$$

Once again, $w,x,y,z = \pm 1$. And it is easily seen to marginalize correctly. For example for $P_{QS\to 1,1}$,

$= \frac{1}{16}(1 - p\,(\chi_{QS}+\chi_{QT}+\chi_{RS}+\chi_{RT}-\chi_{QR}-\chi_{ST}))$

$+ \frac{1}{16}(1 - p\,(\chi_{QS}-\chi_{QT}+\chi_{RS}-\chi_{RT}-\chi_{QR}+\chi_{ST}))$

$+ \frac{1}{16}(1 - p\,(\chi_{QS}+\chi_{QT}-\chi_{RS}-\chi_{RT}+\chi_{QR}-\chi_{ST}))$

$+ \frac{1}{16}(1 - p\,(\chi_{QS}-\chi_{QT}-\chi_{RS}+\chi_{RT}+\chi_{QR}+\chi_{ST}))$

$= \frac{1}{4}(1 - p\,(\chi_{QS}))$

The quantity $(*)$ can be shown to not exceed 2 (Lemma 1). So, the values of the global section remain positive for $p \le \frac{1}{2}$. This corresponds to a classical correlation between measurement outcomes for the state in this range of p.

LEMMA 1 Given unit vectors $\vec{q}, \vec{r}, \vec{s}, and \vec{t}$, the quantity $\vec{q}\cdot\vec{s} + \vec{q}\cdot\vec{t} + \vec{r}\cdot\vec{s} + \vec{r}\cdot\vec{t} - \vec{q}\cdot\vec{r} - \vec{s}\cdot\vec{t}$ cannot exceed 2.

Lemma 1 $0 \le (\vec{q}+\vec{r}-\vec{s}-\vec{t})^2$

$0 \le ((\vec{q}+\vec{r})-(\vec{s}+\vec{t}))^2$

$0 \le (\vec{q}+\vec{r})^2 - 2(\vec{q}+\vec{r})(\vec{s}+\vec{t}) + (\vec{s}+\vec{t})^2$

$0 \le \vec{q}^2 + \vec{r}^2 + 2(\vec{q}\cdot\vec{r}) - 2(\vec{q}\cdot\vec{s}) - 2(\vec{q}\cdot\vec{t}) - 2(\vec{r}\cdot\vec{s}) - 2(\vec{r}\cdot\vec{t}) + \vec{s}^2 + \vec{t}^2 + 2(\vec{s}\cdot\vec{t})$

$0 \le -(\vec{q}\cdot\vec{s}) - (\vec{q}\cdot\vec{t}) - (\vec{r}\cdot\vec{s}) - (\vec{r}\cdot\vec{t}) + (\vec{q}\cdot\vec{r}) + (\vec{s}\cdot\vec{t}) + \frac{\vec{q}^2+\vec{r}^2+\vec{s}^2+\vec{t}^2}{2}$

$(\vec{q}\cdot\vec{s}) + (\vec{q}\cdot\vec{t}) + (\vec{r}\cdot\vec{s}) + (\vec{r}\cdot\vec{t}) - (\vec{q}\cdot\vec{r}) - (\vec{s}\cdot\vec{t}) \le 2$

$\therefore (*) \le 2$

4 Conclusion

Via the Peres-Horodecki condition the non-separability of the state is determined. The classical nature of the measurement correlations is proved via explicit construction of an analytic expression describing the correlations, which is shown to be non-negative (classical) for another range of values. The parametrized ensemble ρ_W is thus shown to be definitively entangled for a range of values overlapping the range of values in which it behaves classically. This demonstrates that entanglement, defined as non-separability, doesn't necessarily display non-classical behavior.

References

[1] Reinhard F Werner. Quantum states with einstein-podolsky-rosen correlations admitting a hidden-variable model. *Physical Review A*, 40(8):4277, 1989.

[2] Samson Abramsky and Adam Brandenburger. The sheaf-theoretic structure of non-locality and contextuality. *New Journal of Physics*, 13(11):113036, 2011.

[3] Michael A Nielsen and Isaac L Chuang. *Quantum computation and quantum information*, volume 546. 2000.